Δ-spaces X and distinguished spaces $C_p(X)$

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Definition 1. *A subset of reals $X \subset \mathbb{R}$ is said to be a* $\underline{\Delta\text{-set}}$ *if X has the following* property $\underline{\Delta}$: *for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X, also with empty intersection, and such that $D_n \subset V_n$ for every $n \in \omega$.*

The existence of an uncountable Δ-set $X \subset \mathbb{R}$ is one of the fundamental set-theoretical problems; it is independent of ZFC (due to results of F. Hausdorff and M.E. Rudin).

Definition 2. *A locally convex space (lcs) E is called* distinguished *if the strong dual of E (i.e. the topological dual of E endowed with the strong topology) is barrelled* (A. Grothendieck).

What two defined above notions from two completely different areas, one from Set Theory, another from Functional Analysis, can have in common?

By $C_p(X)$ we mean the space of all real-valued continuous functions on a Tychonoff topological space X endowed with the topology of pointwise convergence.

Theorem 3. [1] *Let X be a Tychonoff space. A lcs $C_p(X)$ is distinguished if and only if for each $f \in \mathbb{R}^X$ there is a bounded $B \subset C_p(X)$ such that f belongs to the closure of B in $\mathbb{R}^X$.*

We say that a Tychonoff space X is a $\underline{\Delta\text{-space}}$ if X satisfies property Δ, as in Definition 1 above. Quite recently we have shown that the notion of Δ-spaces plays a key role in the study of distinguished C_p-spaces [2].

Characterization Theorem 4. [2] *Let X be a Tychonoff space. A lcs $C_p(X)$ is distinguished if and only if X is a Δ-space.*

My talk will be devoted to the main results published in the joint with J. Kąkol papers [1], [2], [3]; and also my works [4], [5].

References

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[4] A. Leiderman, V.V. Tkachuk, *Pseudocompact Δ-spaces are often scattered*, Monatsh. Math. **197** (2022), 493–503.

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